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ON CERTAIN EXTENSIONS OF VALUED FIELDS

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ABSTRACT. Let $v = v_1 \circ v_2 \circ ... \circ v_n$ be a valuation of a field K with $rankv = n$. Let $(L, z)/(K, v)$ be a finite extension of valued fields where $z = z_1 \circ z_2 \circ ... \circ z_n$ is the extension of v to field L . In this paper it is shown that, if $(L, z)/(K, v)$ is a tame extension then finite extensions of valued fields $(L, z_1)/(K, v_1)$ and $(k_{z_{i-1}}, z_i)/(k_{v_{i-1}}, v_i)$ are tame extensions for $i = 2, ..., n$. In this paper a residual transcendental extension of $w = w_1 \circ w_2 \circ ... \circ w_n$ to $K(x)$ is studied and a characterization of lifting polynomials is given where w_i is the residual extension of v_i for $i = 1, 2, ..., n$.

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1. INTRODUCTION

Throughout this paper, v is a Henselian valuation of K with value group G_v, residue field k_v, valuation ring O_v and $\overline{v}$ is the fixed extension of v to the algebraic clousure $\overline{K}$ of K. For any λ in the valuation ring of v, λ_v^* will denote its $v-$residue i.e. the image of λ under the canonical homomorphism from the valuation ring of v onto residue field k_v. If λ is not an element of valuation ring then for an element $b \in K$ such that $v(\lambda) = v(b)$, λ_v^* will denote the $v-$residue of λ/b. Let L be a finite extension of K, w be an extension of v to L, $e(w/v) = [G_w : G_v]$ and $f(w/v) = [k_w : k_v]$ be the ramification index and the residue degree of w/v respectively. The extension of valued fields $(L, w)/(K, v)$ is called tame extension if the following conditions are satisfied.

i) $[L : K] = e(w/v).f(w/v)$

ii) k_w is a seperable extension of k_v

iii) $e(w/v)$ is not divisible by the characteristic of k_v.

An element $(a, \delta) \in \overline{K} \times G_{\overline{v}}$ is called minimal pair with respect to (K, v) if every $b \in \overline{K}$, $\overline{v}(a - b) \geqslant \delta$ implies $[K(a) : K] \leqslant [K(b) : K]$. Let $K(x)$ be a field of rational functions of one variable over K. The valuation w on $K(x)$ is called a residual transcendental extension of v if k_w is a transcendental extension of k_v. Let w be a residual transcendental extension of v to $K(x)$. Then w is defined by minimal pairs $(a, \delta) \in \overline{K} \times G_{\overline{v}}$. Let $f(x) = Irr(a, K)$ be minimal polynomial of a over K and $\gamma = \sum_{a'} \inf(\delta, \overline{v}(a - a'))$ where a' runs over all roots of $f(x)$. If $F(x) \in K[x]$ and $F(x) = \sum_i F_i(x) f(x)^i$, $\deg F_i < \deg f$ is the $f-$expansion of $F(x)$ then the valuation w on $K(x)$ is defined as $w(F) = \inf_i (\overline{v}(F_i(a)) + i.\gamma)$. Let e be the smallest positive integer such that $e.\gamma \in G_{v_a}$ and $h(x) \in K[x]$ such that $\deg h < \deg f$, $\overline{v}(h(a)) = e.\gamma$ where v_a is the restriction of $\overline{v}$ to field $K(a)$. Let $r = \frac{f^e}{h} \in K(x)$ such that $w(r) = 0$. So r^* is transcendental over residue field k_v and also $k_w = k_{v_a}(r^*)$ is the residue field of w. We will denote r^* by Y. Let $g(Y) \in k_{v_a}[Y]$. $G(x) \in K[x]$ is a lifting of the polynomial $g(Y)$ with respect to w if the following conditions are satisfied

i) $\deg G = e.\deg g.\deg f$

ii) $w(G) = e.\deg g.\gamma$

iii) $\left(\frac{G}{h^{\deg g}}\right)^*_w = g$.

Let c be a root of $G(x)$. It is known that (c, a) is a distinguished pair with respect to v_1. Distinguished pairs, distinguished chains and their relations with lifting polynomials were firstly studied by Popescu and Zaharescu in 1995 [5]. The constants of an algebraic element are important for defining extensions of v to rational function field $K(x)$, giving characterizations of tame extensions and using in the definition of distinguished pairs. Let (K, v_1) be a Henselian valued field, v_i be a valuation of residue field $k_{v_{i-1}}$ for $i = 2, ..., n$ and $v = v_1 \circ v_2 \circ ... \circ v_n$ be composite of valuations $v_1, v_2, ..., v_n$. In this paper tame extensions of a valued field with a valuation v of $rankv = n$ are studied and the definition of the residual transcendental extension w of v to $K(x)$ is shown by using a root of a lifting polynomial. Also it is studied on the lifting of the polynomials with respect to $w = w_1 \circ w_2 \circ ... \circ w_n$ by using the liftings with respect to w_i with $rankw_i = 1$ for $i = 2, ..., n$.

2. MAIN RESULTS

Theorem 2.1. *Let (K, v_1) be a Henselian valued field, v_i be a valuation with residue field k_{v_i} for $i = 2, ..., n$ and $v = v_1 \circ v_2 \circ ... \circ v_n$ be a composite of valuations $v_1, v_2, ..., v_n$. Let L/K be a finite extension, z_1 be an extension of v_1 to L, z_i be an extension of v_i to k_{z_i} for $i = 2, ..., n$ and $w = w_1 \circ w_2 \circ ... \circ w_n$ be an extension of v to L which is a composite of valuations $z_1, z_2, ..., z_n$. We suppose that $(e_i, e_j) = 1$, for $i \neq j$ where e_i is the ramification index of the extension z_i/v_i for $i = 1, ..., n$. If $(L, z)/(K, v)$ is a tame extension then $(L, z_1)/(K, v_1)$ and $(k_{z_{i-1}}, z_i)/(k_{v_{i-1}}, v_i)$ are tame extensions for $i = 2, ..., n$.*

Proof. Let e and f be ramification index and residue degree of the extension $(L, z)/(K, v)$ respectively. Assume that f_i is the residue degree of the extension z_i/v_i for $i = 1, ..., n$. It is clear that $k_{z_n} = k_z$, $k_{v_n} = k_v$ and $f = f_n$. Since $(L, z)/(K, v)$ is a tame extension, the following conditions are satisfied.

i) $[L : K] = e.f$

ii) e is not divisible by the characteristic of the residue field k_v.

iii) k_z is a seperable extension of k_v.

Since $e = [G_z : G_v]$, for an element $b \in L$, e is the smallest positive integer such that $e.z(b) \in G_v$. Let $b_i = b^*_{i-1}$ for $i = 1, ..., n-1$ where $b_1 = b^*$. Keeping in view of the definition of composite valuation we have

$$e.z(b) = e.(z_1(b), z_2(b_1), ..., z_n(b_{n-1})) \in G_v = G_{v_1} \times G_{v_2} \times ... \times G_{v_n}.$$

Consequently,

$$e.z_1(b) \in G_{v_1},\ e.z_2(b_1) \in G_{v_2}, ..., e.z_n(b_{n-1}) \in G_{v_n}.$$

Therefore $e_i \,|\, e$ for $i = 1, ..., n$. So by the hypothesis; it is seen that $e = e_1.e_2....e_n$. If this equality and the equality $f = f_n$ are written in the first condition (i) then it is obtained that

$$[L : K] = e.f = e_1.e_2.....e_n.f_n. \tag{1}$$

The degree of the extension $(k_{z_{n-1}}, z_n)/(k_{v_{n-1}}, v_n)$ is $f_{n-1} = e_n.f_n.d_n$ where d_n is the defect of this extension. If the equality $\frac{f_{n-1}}{d_n} = e_n.f_n$ is written in (1) then

$$[L : K] = e.f = e_1.e_2.....e_n..\frac{f_{n-1}}{d_2}$$

is obtained. The degree of the extension $(k_{z_{n-2}}, z_{n-1})/(k_{v_{n-2}}, v_{n-1})$ is $f_{n-2} = e_{n-1}.f_{n-1}.d_{n-1}$ where d_{n-1} is the defect of this extension. By continuing in the same way, we have

$$e.f = e_1.f_1.d_1 = e_1.f_1.\frac{1}{d_2}.\frac{1}{d_3}....\frac{1}{d_n}.$$

Therefore we obtain;

$$d_1 = \frac{1}{d_2}.\frac{1}{d_3}.....\frac{1}{d_n}.$$

Hence $d_2 = d_3 = ... = d_n = 1$ since $d_i \in Z$ for $i = 1, ..., n$ and so $d_i = 1$. Thus the degree of the extension $(k_{z_{i-1}}, z_i)/(k_{v_{i-1}}, v_i)$ is $f_{i-1} = e_i.f_i$ for $i = 2, ..., n$ and the degree of the extension $(L, z_1)/(K, v_1)$ is $[L : K] = e_1.f_1$. k_{z_n}/k_{v_n} is a seperable extension since $k_{z_n} = k_z$ and $k_{v_n} = k_v$. Keeping in view of the the equality $chark_{v_n} = chark_v$ we obtain

$$chark_v \nmid e \Rightarrow chark_{v_n} \nmid e = e_1.e_2....e_n \Rightarrow chark_{v_n} \nmid e_n.$$

For completing the proof of the theorem, subsituations will be studied.

Case1. If $chark_{v_{n-1}} = 0$ then $k_{v_{n-1}}$ is a perfect field. Therefore $k_{z_{n-1}}/k_{v_{n-1}}$ is a seperable extension and e_{n-1} is not divisible by the characteristic of $k_{v_{n-1}}$.

Case 2. Let $chark_{v_{n-1}}$ be a prime number. Then $k_{v_{n-1}} = k_v$ and $k_{z_{n-1}} = k_z$. Thus $k_{z_{n-1}}/k_{v_{n-1}}$ is a seperable extension and

$$chark_v \nmid e \Rightarrow chark_{v_{n-1}} \nmid e_1.e_2....e_n \Rightarrow chark_{v_{n-1}} \nmid e_{n-1}.$$

By continuing in the same way, it is obtained that k_{z_i}/k_{v_i} is a seperable extension and $chark_{v_i} \nmid e_i$ for $i = n-2, n-1, ..., 2, 1$. Consequently it is obtained that the extensions $(L, z_1)/(K, v_1)$ and $(k_{z_{i-1}}, z_i)/(k_{v_{i-1}}, v_i)$ are tame extensions for i = 2, . . . ,n . □

Let (K, v_1) be a henselian valued field, v_i be a valuation of residue field k_{v_i} for $i = 2, ..., n$ and $v = v_1 \circ v_2 \circ ... \circ v_n$ be composite of valuations $v_1, v_2, ..., v_n$. Let w_1 be a residual transcendental extension of v_1 to the rational function field $K(x)$ defined by minimal pair (a_1, δ_1), w_i be a residual transcendental extension of v_i to the residue field $k_{w_{i-1}}$ defined by minimal pair (a_i, δ_i) for $i = 2, ..., n$ and $w = w_1 \circ w_2 \circ ... \circ w_n$ be an extension of v to $K(x)$. Let $f_1(x) = Irr(a_1, K)$ be a minimal polynomial of a_1 respect to K, $w_1(f_1) = \gamma_1$ and e_1 is the smallest positive integer such that $e_1\gamma_1 \in G_{v_{a_1}}$ where v_{a_1} is the restriction of $\overline{v_1}$ to $K(a_1)$ and $h_1(x) \in K[x]$ such that $\deg h_1 < \deg f_1$, $v_{a_1}(h_1(a_1)) = e_1\gamma_1$. Let $f_i = Irr(a_i, k_{v_{i-1}})$ be a minimal polynomial of a_i respect to $k_{v_{i-1}}$, $w_i(f_i) = \gamma_i$ and e_i is the smallest positive integer such that $e_i\gamma_i \in G_{v_{a_i}}$ where v_{a_i} is the restriction of $\overline{v_i}$ to the field $k_{v_{i-1}}(a_i)$ and $h_i(x) \in k_{v_{a_{i-1}}}[Y_{i-1}]$ such that $\deg h_i < \deg f_i$, $v_{a_i}(h_i(a_i)) = e_i\gamma_i$ where w_1-residue of ${f_1}^{e_1}/h_1$ is Y_1 and w_i-residue of ${f_i}^{e_i}/h_i$ is Y_i for $i = 2, ..., n$. Under the above notations we have the following theorems.

Theorem 2.2. *Let $B_{i-1}(Y_{i-1}) \in k_{v_{a_{i-1}}}[Y_{i-1}]$ be a lifting polynomial of $B_i(Y_i) \in k_{v_{a_i}}[Y_i]$ with respect to w_i for $i = 2, ..., n$ and $B(x) \in K[x]$ be a lifting polynomial of $B_1(Y_1) \in k_{v_{a_1}}[Y_1]$ with respect to w_1. Then $B(x)$ is a lifting polynomial of $B_n(Y_n) \in k_{v_{a_n}}[Y_n]$ with respect to $w_1 \circ w_2 \circ ... \circ w_n$.*

Proof. Proof can be obtained by induction. For $n = 2$, proof is clear from [8]. Suppose that assertion is true for $n-1$. In other words, we assume that $B(x)$ is a lifting polynomial of $B_{n-1}(Y_{n-1})$ with respect to $w_1 \circ w_2 \circ ... \circ w_{n-1}$. Using the definition of lifting polynomial

$$\deg B = \deg B_{n-1}. \deg g_{n-1}, \tag{2}$$

$$(w_1 \circ w_2 \circ ... \circ w_{n-1})(B) = \deg B_{n-1}.(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1}), \tag{3}$$

$$\left(\frac{B}{H_{n-2}^{\deg B_{n-1}}}\right)^*_{w_1 \circ w_2 \circ ... \circ w_{n-1}} = B_{n-1} \tag{4}$$

are satisfied. Since B_{n-1} is a lifting polynomial of B_n with respect to w_n

$$\deg B_{n-1} = \deg B_n. \deg f_n, \tag{5}$$

$$w_n(B_{n-1}) = \deg B_n.\gamma_n, \tag{6}$$

$$\left(\frac{B_{n-1}}{h_n^{\deg B_n}}\right)^*_{w_n} = B_n \tag{7}$$

are hold. Writing the equality (5) in the equation (2) it is obtained that $\deg B = \deg B_n. \deg g_n$. Considering the equations (3), (5) together with the equation (6) it is obtained that

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_{n-1})(B) &= (w_1(B), w_2(B_1), ..., w_{n-1}(B_{n-2}))\\ &= \deg B_{n-1}.(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1})\\ &= \deg B_n. \deg f_n.(e_1. \deg f_{n-1}. \deg f_{n-2}...\\ &\quad \deg f_2.\gamma_1,\ ...,\ \deg f_{n-1}.\gamma_{n-2}, \gamma_n).\end{aligned}$$

Then

$$(w_1 \circ w_2 \circ ... \circ w_n)(B) = \deg B_n.(w_1 \circ w_2 \circ ... \circ w_n)(g_n)$$

is satisfied. Considering the equalities

$$(w_1 \circ w_2 \circ ... \circ w_n)(g_n) = (\overline{v_1} \circ \overline{v_2} \circ ... \circ \overline{v_n})(H_{n-1}(c_{n-1})) = (w_1 \circ w_2 \circ ... \circ w_n)(H_{n-1})$$

from [5] and

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_n)(g_n) &= (\deg f_n.e_1. \deg f_{n-1}. \deg f_{n-2}...\\ &\quad \deg f_2.\gamma_1, ...\ ,\ \deg f_n.\gamma_{n-1}, \deg B_n.\gamma_n)\end{aligned}$$

the equality

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_{n-1})(H_{n-1}) &= \deg f_n.(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1})\\ &= \deg f_n.(w_1 \circ w_2 \circ ... \circ w_{n-1})(H_{n-2})\end{aligned}$$

is obtained. Since $w_n(h_n) = \gamma_n$ it is seen that

$$\left(\frac{H_{n-1}}{H_{n-2}^{\deg f_n}}\right)^*_{w_1 \circ w_2 \circ ... \circ w_{n-1}} = h_n \tag{8}$$

Using the equalities (4), (5), (7) and (8) it is obtained that

$$\left(\frac{B}{H_{n-1}^{\deg B_n}}\right)^*_{w_1 \circ w_2 \circ ... \circ w_n} = B_n$$

So the proof is completed. □

Theorem 2.3. *Let $g_i \in K[x]$ be a lifting polynomial of $f_i \in k_{v_{a_{i-1}}}[Y_{i-1}]$ with respect to $w_1 \circ w_2 \circ ... \circ w_{i-1}$ for $i = 2, ..., n$. Then the equality*

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_n)(g_n) \;=\; & (e_1. \deg f_n. \deg f_{n-1}.... \deg f_2.\gamma_1,\\ & \deg f_n. \deg f_{n-1}..... \deg f_3.\gamma_2, ...,\\ & \deg f_n. \deg f_{n-1}..... \deg f_3.\gamma_2, ...,\\ & \deg f_n. \deg f_{n-1}.\gamma_{n-2}, \deg f_n.\gamma_{n-1}, \gamma_n)\end{aligned}$$

is satisfied.

Proof. Proof can be obtained by induction. For $n = 2$, $(w_1 \circ w_2)(g_2) = (\deg f_2.\gamma_1, \gamma_2)$ is known from [8]. Suppose that assertion is true for $n - 1$. In other words, we assume that the equality

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1}) \;=\; & (e_1. \deg f_{n-1}. \deg f_{n-2}.... \deg f_2.\gamma_1,\\ & \deg f_1. \deg f_{n-2}..... \deg f_3.\gamma_2, ...,\\ & \deg f_{n-1}.\gamma_{n-2}, \gamma_{n-1})\end{aligned}$$

is satisfied. The composite valuation $w_1 \circ w_2 \circ ... \circ w_{i-1}$ is defined by the minimal pair (c_{n-2}, λ_{n-2}) where c_{n-2} is a root of the polynomial g_{n-1} and $\lambda_{n-2} = (\delta_1, \delta_2, ..., \delta_{n-1})$ from [4]. Let $v_{c_{n-2}}$ denote the restriction of the valuation $\overline{v}$ to the field $K(c_{n-2})$ and e be the smallest positive integer such that $e.(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_n) \in G_{v_{c_{n-2}}}$. The equality

$$(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_n) = e.(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1}). \deg f_n$$

is satisfied since g_n is a lifting of the polynomial f_n with respect to composite valuation $w_1 \circ w_2 \circ ... \circ w_{n-1}$. We suppose that f_n is a lifting of some polynomial $J_n \in k_{v_{a_n}}[Y_n]$. Thus $\deg f_n = e_n. \deg J_n. \deg f_n$ and so $e_n = 1$. Also $e = 1$ is obtained with the same way in [8]. Hence

$$(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_n) = (w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1}). \deg f_n$$

is written. Keeping in view of the equation

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_{n-1})(g_n) \;=\; & (w_1 \circ w_2 \circ ... \circ w_{n-1})(g_{n-1}). \deg f_n,\\ & (e_1. \deg f_n \deg f_{n-1}. \deg f_{n-2}.... \deg f_2.\gamma_1,\\ & \deg f_n \deg f_1. \deg f_{n-2}..... \deg f_3.\gamma_2, ...,\\ & \deg f_n \deg f_{n-1}.\gamma_{n-2}, \deg f_n \gamma_{n-1})\end{aligned}$$

and the fact that g_n is a lifting of the polynomial f_n with respect to composite valuation $w_1 \circ w_2 \circ ... \circ w_{n-1}$ it is obtained that

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_n)(g_n) &= (e_1.\deg f_n \deg f_{n-1}.\deg f_{n-2}....\deg f_2.\gamma_1,\\ &\quad \deg f_n \deg f_1.\deg f_{n-2}.....\deg f_3.\gamma_2, ...,\\ &\quad \deg f_n \deg f_{n-1}.\gamma_{n-2}, \deg f_n \gamma_{n-1}, w_n(f_n))\\ &= (e_1.\deg f_n.\deg f_{n-1}....\deg f_2.\gamma_1,\\ &\quad \deg f_n.\deg f_{n-1}.....\deg f_3.\gamma_2, ...,\\ &\quad \deg f_n.\deg f_{n-1}.....\deg f_3.\gamma_2, ...,\\ &\quad \deg f_n.\deg f_{n-1}.\gamma_{n-2}, \deg f_n.\gamma_{n-1}, \gamma_n)\end{aligned}$$

as desired. □

Theorem 2.4. *Let $g_i \in K[x]$ be a lifting polynomial of $f_i \in k_{v_{a_{i-1}}}[Y_{i-1}]$ with respect to $w_1 \circ w_2 \circ ... \circ w_{i-1}$ for $i = 2, ..., n$ and c_{n-1} be a root of g_n that defines composite valuation $w = w_1 \circ w_2 \circ ... \circ w_n$ with $\lambda_{n-1} = (\delta_1, \delta_2, ..., \delta_n)$ and v. If $F(x) \in K[x]$ and $F(x) = \sum_i F_i g_n(x)^i$, $\deg F_i < \deg g_n$ is the g_n-expansion of F then the valuation w is defined as*

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_n)(F) &= \inf_i \{(\overline{v_1}(F_i(c_{n-1})), w_2((F_i(x)/F_i(c_{n-1}))^*_{w_1}),\\ &\quad w_3((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2}), ...,\\ &\quad w_n((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2 \circ ... \circ w_n})) +\\ &\quad i.(e_1.\deg f_n.\deg f_{n-1}....\deg f_2.\gamma_1,\\ &\quad \deg f_n.\deg f_{n-1}.....\deg f_3.\gamma_2, ...,\\ &\quad \deg f_n.\deg f_{n-1}.\gamma_{n-2}, \deg f_n.\gamma_{n-1}, \gamma_n)\end{aligned}$$

Proof. It is known that the valuation $w_1 \circ w_2 \circ ... \circ w_{n-1}$ is defined by the minimal pair (c_{n-2}, λ_{n-2}) where c_{n-2} is a root of the polynomial g_{n-1} and $\lambda_{n-2} = (\delta_1, \delta_2, ..., \delta_{n-1})$ from [4]. If we consider that g_n is a lifting of the polynomial f_n with respect to $w_1 \circ w_2 \circ ... \circ w_{n-1}$ and the composite valuation $w = w_1 \circ w_2 \circ ... \circ w_n$ is defined by the mininmal pair (c_{n-1}, λ_{n-1}) where c_{n-1} is a root of the polynomial g_{n-2} and $\lambda_{n-1} = (\delta_1, \delta_2, ..., \delta_{n-2})$ then we see that $(w_1 \circ w_2 \circ ... \circ w_n)(F_i) = (\overline{v_1} \circ \overline{v_2} \circ ... \circ \overline{v_n})(F_i(c_{n-1})))$. Hence the equality $(w_1 \circ w_2 \circ ... \circ w_j)(F_i) = (\overline{v_1} \circ \overline{v_2} \circ ... \circ \overline{v_j})(F_i(c_{n-1})))$ is hold for $j = 1, ..., n$. So $\frac{F_i(x)}{F_i(c_{n-1})}$ have a residue with respect to $w_1 \circ w_2 \circ ... \circ w_j$. Therefore we write the equality

$$\begin{aligned}(w_1 \circ w_2 \circ ... \circ w_n)(F_i) &= (\overline{v_1}(F_i(c_{n-1})), w_2((F_i(x)/F_i(c_{n-1}))^*_{w_1}),\\ &\quad w_3((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2}), ...,\\ &\quad w_n((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2 \circ ... \circ w_n}))\end{aligned}$$

by using the definition the composite valuation. Keeping in view of the expression from Teorem 2.3. we obtain that

$$\begin{aligned}(w_1 \circ ... \circ w_n)(F) &= \inf_i \{(w_1 \circ ... \circ w_n)(F_i) + i.(w_1 \circ ... \circ w_n)(g_n)\} \\ &= \inf_i \{(\overline{v_1}(F_i(c_{n-1})), w_2((F_i(x)/F_i(c_{n-1}))^*_{w_1}), \\ &\quad w_3((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2}), ..., \\ &\quad w_n((F_i(x)/F_i(c_{n-1}))^*_{w_1 \circ w_2 \circ ... \circ w_n})) + \\ &\quad i.(e_1. \deg f_n. \deg f_{n-1} .... \deg f_2.\gamma_1, \\ &\quad \deg f_n. \deg f_{n-1} ..... \deg f_3.\gamma_2, ..., \\ &\quad \deg f_n. \deg f_{n-1}.\gamma_{n-2}, \deg f_n.\gamma_{n-1}, \gamma_n).\end{aligned}$$

So the proof is completed. □

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